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Equivalent circuit for transient conduction and convection systems

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EQUIVALENT CIRCUIT FOR TRANSIENT CONDUCTION AND CONVECTION SYSTEMS.

by

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ABSTRACT

Transient response of a natural convection system is investigated by numerical simulation using FLUENT code. An Integrator Circuit analogy was recently proposed for natural convection system. The proposed analogy was further confirmed by these recent simulations. New simulation results also suggest that a natural convection system acts as a "Low Pass" filter for transients. Transient transmission/ attenuation factor was found to be a function of both fluid properties and the flow characteristics. Transmission/attenuation factor was also found to be a strong function of fluctuation frequency. These results may prove a significant design tool for Gen IV natural convection system particularly for LFR.

Explicit formulas for the elements of the inverse of tridiagonal matrices are developed. The formulas are recursive and applicable to symmetric, non-symmetric, equal and non-equal coefficient matrices. For the case of a general tridiagonal matrix four recursive formulas are developed to obtain the elements of the matrix inverse. All formulas are deduced based on the fact that a matrix multiplied by its inverse result in a unit matrix. Also three special cases are discussed. First is the inverse of a symmetric tridiagonal matrix. Second is the inverse of a tridiagonal Toeplitz symmetric matrix. Third is the inverse for a general constant diagonal matrix (Toeplitz). These results are very helpful in economical simulation of reactor physics, heat and mass transport, electrical circuit solution, fluid flow and many other engineering problems.

ACKNOWLEDGMENTS

All thanks are due to ALLAH (God) and may His peace and blessings be upon the last of the messengers and prophets prophet Mohammad. It was only through ALLAH's help and support that this work was accomplished. I can never thank Him enough for all the blessing He bestowed upon me throughout my life and I ask His forgiveness of all my shortcomings.

As is always the case, there have been many people who have had an influence on the work that is contained in this thesis. I would like to especially recognize my advisor, Dr. Shoaib Usman. It is certainly through his guidance and encouragement that this work is complete. He has never been one to insist on anything to be done in a certain way but has always been there to provide guidance. Dr. Usman was always ready to give advice in anytime. Particular thanks to Dr. Abdallah Shaaban for all his help throughout my research. Also, my deepest appreciation to Dr. Gary Mueller and Dr. Seungjin Kim for their time and effort as committee members.

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Finally, there is a single person who has had a greater effect on this work than anyone else, and that is my wife. She has given me total support and encouragement, taking care of family duties to provide me with the environment and the time necessary to complete this effort.

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NATURAL CONVECTION'S TRANSIENT BEHAVIOR

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Transient response of a natural convection system is investigated by numerical simulation using FLUENT code. An Integrator Circuit analogy was recently proposed for natural convection system. The proposed analogy was further confirmed by these recent simulations. New simulation results also suggest that a natural convection system acts as a "Low Pass" filter for transients. Transmission characteristics of natural convection system were investigated using sinusoidal temperature at the source side boundary. Transient transmission factor was found to be a function of both fluid properties and the flow characteristics. Transmission factor was also found to be a strong function of fluctuation frequency. These results may prove a significant design tool for Gen IV natural convection system particularly for LFR or Molten Salt Reactors or Molten Salt Reactor.

KEYWORDS: *Natural Convection, Heat Transfer, Gen IV Nuclear Reactors, Circuit Analogy.*

I. INTRODUCTION

Natural convection is a very important phenomenon for a number of engineering systems including nuclear reactors. For this reason the phenomenon has been an area of significant studies. For present day nuclear system, natural convection is relied upon for cool down and decay heat removal[1]. Likewise, new CANDU-X would also incorporate enhanced passive natural circulation for heat removals to ensure safety[2]. This renewed interest has resulted in a number of recent studies[3,4,5,6].

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For the next generation of nuclear reactors, use of natural convection is even more significant. For example, natural convection is the primary mode of heat transfer for Lead-Cooled Fast Reactor systems (LFR)[7]. There are several other aspects of natural convection that may potentially impact the performance of LFR. For example, a recent study reported an enhancement of oxygen transfer in liquid lead and lead bismuth eutectic by natural convection[8]. Oxygen concentration control is found to be a critical factor for corrosion control in LFR's internals.

For certain systems transition between natural and forced convection is very important. Skreba and co-workers studied this transition at a research reactor[9]. All these studies indicate that the phenomenon of natural convection is extremely critical for safety and normal operation of present and future generation of nuclear reactor. This paper presents a simple electric circuit analogy based model for natural convection which may be helpful in analysis of transient behavior of a natural convection system. The proposed analysis may provide a quick tool for modeling of various new reactor concepts.

A very useful analogy between steady state conductive heat transfer and simple resistor circuits was developed[10]. Temperature gradient across a layer of pure conductive material, according to this analogy acts as the voltage difference across a resistor. Thermal resistance/insulation of the material is analogous to the electrical resistance of a resistor. In this manner, a steady state heat transfer problem can be transformed into an equivalent electrical circuit and solved using standard techniques for solving electrical circuits. Fig. 1 shows an equivalent conduction circuit. Where Δx is the layer thickness, A is the cross-sectional area and k is the thermal conductivity of the material.

However, this simple circuit analogy is not capable of modeling the transient response of a thermal system. This limitation is due to the fact that there is no means to incorporate the thermal inertia of various layers of the materials being used as insulators/resistors. In order to incorporate this thermal inertia, modification to the resistor circuit has been proposed which lead to the development of a "Lumped Model[11]". Lumped models represent the heat capacity of the material in terms of an equivalent capacitor in the analogous circuit. This allows for a transient analysis of the system. In order to closely represent the physical configuration three combinations of resistor and capacitors are generally used. These combinations are; T-Lumping, L-Lumping and Π Lumping. Fig. 2 shows these lumping arrangements.

Since no transient analysis was possible with a simple resistor circuit, lumped system analogy is a major step forward in facilitating transient analysis of complex system involving pure conduction. Lumped analogy has gained considerable popularity. Characteristic time constant for an equivalent circuit for a lumped system is known to be inversely proportional to thermal diffusivity (α_{Mol})[11].

Recently Integrator Circuit (RC-circuit) was proposed as an analogy for natural convection which is a logical extension of Lumped Model[12,13]. Conductive heat transfer system is represented in terms of equivalent resistor(s) and heat capacity is represented by equivalent capacitor(s). An analogous circuit is shown in Fig. 3. It is well known that, in the case of convective heat transfer, thermal resistance is significantly reduced as compared to pure conduction. Moreover, we noted that the thermal capacity (capacitance of the equivalent circuit) of the system to store energy is also augmented by kinetic energy of the fluid. Therefore, the energy supplied to the system is stored in form of the thermal energy of the fluid as well as the kinetic energy of the fluid which is set in motion due to the phenomenon of natural convection. This aspect of convection, i.e. change in capacitance due to fluid motion, has not been studied thus far.

II. CIRCUIT ANALOGY FOR HEAT TRANSFER

Our initial experimental results and numerical simulations[12] suggested that a natural convection system can be modeled by an integrator circuit as shown in Fig. 3. This analogy is a logical extension of the circuit analogy for conduction. In the case of pure conduction the resistance, R would be equal to the thermal resistance of the fluid layer. And, the capacitance, C would mean the thermal capacity of the fluid. Therefore, the characteristic time of the system can be written as;

$$\tau = R.C \quad . \tag{1}$$

Both thermal resistance and capacitance of a fluid are intrinsic properties of the material. Therefore the time response of a pure conduction system is known to depend only on the physical properties of the fluid and the geometry of the heat transfer system.

Unlike conduction, behavior of a convection system depends both on the fluid and the flow characteristics. Therefore, the equivalent conductivity of a convection system is the net sum of the conductivity offered by the conduction and the conductivity due to convection. Likewise, the net capacitance of the system would be the aggregate of the thermal capacitance of the material (fluid) plus the K.E. capacitance of the system; i.e., the system ability to store energy in form of kinetic energy of the fluid set in motion due to natural convection. Based on the present study we propose that a natural convection can also be represented by an equivalent RC circuit (lumped model) as shown in Fig. 3. However, the equivalent resistance and capacitance of a natural convection system would be significantly different from the resistance and capacitance of a pure conduction system.

II.A. Characteristics of an Integrator Circuit

Basic characteristics of an integrator (RC) circuit are well known. First of all, it is well known that if a step voltage is applied to the circuit, as shown in Fig. 3, the output voltage will build up exponentially[14] as given by Equation (2);

$$V_{out} = V_{In}(1-e^{-t/\tau}) \tag{2}$$

Secondly, if the input voltage varies in the form of a periodic function, the circuit will act as a low pass filter[9], that means high frequency fluctuations will be filtered out while low frequency fluctuations will pass through the system. This property of an integrator circuit is routinely utilized in the field of signal processing to design frequency domain filters.

Thirdly, when the input voltage follows a sinusoidal function, the output voltage will also follow the sinusoidal function with same frequency of oscillations as the input voltage. However, the amplitude of oscillations at the output will be reduced. The decrease in the amplitude depends on; the resistance, and the capacitance of the circuit and the frequency of oscillations. As the frequency of oscillations increase, the amplitude of the oscillations at the output side decreases.

Lastly, the characteristic time constant of an integrator circuit doesn't depend on the value of the input voltage. The time constant only depends on the values of electrical capacitance and resistance. The analogy between integrator circuit and the phenomenon of natural convection and how various characteristics of an *RC* circuit manifest for a convection system is the topic of this paper.

II.B. Equivalent Convection Circuit

Our previous work[12,13] demonstrated that the transient response of both conductive and convective heat transfer is analogous to that of an integrator circuit. Both experimental data and numerical simulation for conductive as well as convective heat transfer produced an exponential temperature rise at the sink side in response to a step temperature increase at the source side. The characteristic time constant for convection was however found to be significantly smaller than the conduction heat transfer time constant.

In the present study we investigated the second analogy between an integrator/RC circuit and transient heat transfer (conduction and convection modes), from the point of view of filtering characteristics. It is shown that high frequency temperature fluctuations at the source side do not transmit through the heat transfer system, while low frequency oscillations pass through.

The third analogy between an RC circuit and transient heat transfer (conduction and convection modes) is the attenuation of amplitude at sink side temperature in response to a fluctuating temperature at the source side. This attenuation was observed and extensively investigated. Moreover, we observed that the frequency of temperature oscillation at the sink side is identical to the frequency of temperature fluctuation at the source side. The values of resistance and capacitance in convective and conductive systems are different. This lead to different time constants as discussed in our previous work[12]. In response to an oscillating source side temperature, the temperature amplitude attenuation observed at the sink side should therefore be different for convective and conductive systems even for identical fluid, geometry and source side fluctuation frequency. This observation is analogous to an integrator circuit response. When subjected to same input voltage frequency, the output amplitude of an integrator circuit is dependent only on the values of R and C. Our current work confirms this aspect of the circuit analogy and shows that attenuation for convective system is significantly different from the attenuation for conductive system even for same oscillation frequency at the source side.

Finally, we observed that the characteristic time constant of a heat transfer system is independent of the temperature difference between the source and the sink sides. The time constant of a heat transfer system was found to be dependent on the fluid properties and in the case of convection it also depends on the flow characteristics. All aspects of this circuit analogy for a heat transfer system are extensively investigated using numerical simulation and the results are presented here. These results provide in-sight into the phenomenon of natural convection which is critical for a number of engineering applications including nuclear reactor safety.

Our simulation results are very helpful in developing a simple analytical tool for modeling transient response of a natural convection system. This effort is quite timely, since many advanced nuclear systems are relying on natural convection[5,6]. For example, heat rejection from gas cooled reactor's pressure vessel to the passive cooling system, heat transfer between Lead-cooled Fast Reactors (LFR) core and the super critical carbon dioxide heat exchangers, molten salt cooled reactors and Sodium-cooled Fast Reactor's (SFR) all rely on natural convection for passive decay heat removal.

The strong frequency dependence of transient response is significant for analysis of several GEN IV reactor systems. For example, one very desirable feature of Lead-cooled Fast Reactors (LFR) is autonomous load following. However, it is reported that LFR is unable to follow fast transients[15]. Simulation results presented here provide a plausible explanation for LFR slow response. Analysis of complex natural convection systems requires very expensive CFD modeling codes. One goal of our present study is to develop a modeling tool to study these transients quickly without expensive CFD scheme. We also present our initial simulation results for a Pb-Bi. These results show that the response of Pb-Bi coolant is similar to that of water which has been extensively investigated in this study.

III. CONVECTION AND CONDUCTION SIMULATIONS

In our previous work[12], cylindrical geometry was investigated. Behavior of fluid enclosed between two parallel discs (perfectly circular) was experimentally investigated.

These experimental results were subsequently validated by numerical simulations[12,13]. The proposed analogy between RC circuit and natural convection can be tested for large discs of any shape. The phenomenon is expected to be unaltered as long as the 1D approximation is valid. However, values for R and C will include geometrical parameters of the system. In the presented study we simulated the phenomenon of natural convection for an elliptical layer of fluid and we confirmed that the analogy is valid independent of the shape of the fluid layer.

In the current work, an elliptical layer of fluid with minor axis of 38.1 cm and major axis of 76.2 cm was used. Thickness of the layer was 2 mm. Most of our simulations were for liquid water. We have started simulation for Pb-Bi eutectic alloy and our initial result is reported here. Fig. 4 shows a plot of the simulated elliptical fluid volume.

III.A. Simulation Setup

A very fine grid was generated using Gambit[16]. Rayleigh-Bénard convection apparatus was modeled by dividing the fluid volume into 20 axial layer and 400 circumferential divisions. A part of the grid is shown in Fig. 5. This mesh was imported into FLUENT. FLUENT code was used to simulate 3D flow for unsteady conditions using first order implicit scheme. Small time steps of only 0.05 second were taken in the unsteady state simulation. Typically each simulation took 58 -72 Hrs. of computer time on a Pentium 4 (3.4 GHz) processor with a RAM of 1 GB. Results of these simulations are reported in the following section.

III.B. Boundary Conditions

Circumferential wall of the fluid layer was always perfectly insulated. During the first phase of this study, the sink side wall was also perfectly insulated while a sinusoidal temperature variation (given by equation 3) was applied at the source side;

$$T_{source} = 348 + A_o \sin(\omega t) \tag{3}$$

It was observed that as the temperature of the sink side approaches the temperature of the source side, convection currents die out and conduction prevailed as the sole mode of heat transfer. At this point, and beyond sink side temperature will remain same for both pure conduction and a system which started with convective mode of heat transfer, but later convection died and the system transformed into conduction system. For this reason attenuation of temperature at the sink side was observed to be the same for both conduction and convection systems.

To observe the difference in attenuation between convective and conductive systems we needed to maintain convection currents as the temperature of the sink side approaches the temperature of the source side. Convective boundary conditions were used to ensure that convection current never dies out through out the simulation. For this reason, sink side wall was subjected to the same convection conditions for the case of conduction and convection systems. Convective boundary condition helped to maintain a minimum temperature difference between the source and sink sides and therefore maintain the convection currents as modeling time passes.

For integrator circuit time constant only depends on the resistance and capacitance of the circuit and is independent of the applied step voltage at the input. To simulate the analogous heat transfer system a non convective boundary conditions with step rise in temperature at the source was used.

IV. SIMULATION RESULTS

IV. A. Non Convective Boundary Conditions Results (Sinusoidal Temperature Variations at Source)

Temperature response of the sink side for pure conduction (top plate heating) when the source side temperature oscillates at different frequencies are shown in Fig. 6. It is clear that as the time progresses the sink side (lower plate) temperature starts to fluctuate at the same frequency as the source side (top plate) oscillation. But the amplitude of sink side temperature oscillation is smaller than the source side. For these runs, amplitude of source side fluctuations was chosen to be ± 5 °K. It is clear that as the frequency of fluctuations increase at the source side, the amplitude of temperature oscillation at the sink side is reduced. At a high enough frequency, no temperature fluctuations are observed at the sink side.

Frequency dependence of a convective system was investigated by repeating the simulation for bottom plate heating. Results of these simulations are displayed in Fig. 7.

A comparison of both cases is given in Fig. 8. For the first forty seconds, response of convective system is noticeably different from that of pure conduction. However, as the time progresses, since the sink side wall is insulated the average sink temperature approaches the average source side temperature. Therefore, convection dies out and amplitude of temperature fluctuations at the sink side is observed to be identical to that for pure conduction.

The transmission factor (λ), defined as the ratio of sink side amplitude to source side amplitude given by equation (4);

$$\lambda = \frac{A_{sin\,k}}{A_{source}} * 100 \ (\%) \tag{4}$$

is plotted vs. frequency for both conductive and convective systems in Fig. 9. It is clear that as the frequency increases transmission reduces or the attenuation increase. As can be seen, at a sufficiently high frequency transmission drops to zero and fluctuations at the source side are completely filtered out.

IV. B. Convective Boundary Conditions Results

(Sinusoidal Temperature Variations at Source)

Low frequency temperature fluctuations at the source side are followed by oscillating temperature at the sink side with the same frequency. This behavior is common for both conductive and convective systems. The amplitude of oscillations at the sink side is smaller than the source side and as shown in the previous section, the transmission factor (λ) is a strong function of frequency of oscillation. Moreover, it is well known that convection enhances heat transfer in fluids[6]. Consequently, for a given fluid and frequency of source side oscillation, a convection system is expected to have higher value of transmission factor (λ), than the corresponding conduction system. To ensure that convection remains the mode of heat transfer throughout the simulation a convective boundary condition was used at the sink side. The average temperature at the sink side was maintained low enough such that there was always sufficient temperature difference between sink and source and hence natural convection continued to exist. Fig. 10 shows a typical example of temperature variations at both the sink and source side for the case of convective system (bottom plate heating).

Similar to pure conductive system, for the convective boundary conditions (and hence persistent convection in the enclosure) when the frequency of the fluctuations increases at the source side the amplitude of fluctuations is reduced at the sink side. The transmission factor (λ) is plotted in Fig. 11 for both conductive and convective systems. It is clear that λ in the case of convection is higher than that in the case of conduction. Convective system permits the passage of high frequency oscillations while the conductive system fails to pass those high frequency oscillations. At even higher frequencies the convective system will completely filter the fluctuations.

IV. C. Non Convective Boundary Conditions Results

(Step Temperature Rise at Source)

To demonstrate that the characteristic time constant is independent of the temperature gradient, different temperature gradients were applied and for each case time constant was obtained using equation (5);

$$\theta(at\, t=\tau)=\theta_0\left(1-e^{-\frac{\tau}{\tau}}\right)=0.632\theta_0 \tag{5}$$

As seen in Fig. 12, for a very wide range of ΔT (i.e., temperature gradient applied between the source and sink walls) characteristic time constant is independent of the temperature difference. Slight variation in the time constant is due to statistical uncertainty and approximations introduced during data processing.

IV. D. Lead Bismuth Simulation Results

Use of natural convection for Gen IV (LFR) is very attractive in that, it provides a coupling between the load side of the system and the reactor core (i.e., negative thermal reactivity which controls the rate of fission and power production). This coupling enables autonomous load following. Simulation studies on LFR suggest that the reactor is not very responsive to fast transients[14]. The cooling cycles doesn't respond quickly enough to introduce reactivity changes in the core to follow a fast transient at the generator side. In our current and previous work we studied the mechanism of filtration and time response of convective and conductive systems.

Properties of Lead-bismuth eutectic[17] alloy were imported into the FLUENT code and initial simulations were performed with convective boundary conditions. For the actual reactor system, convective boundary conditions represent the heat removal from the reactor via supercritical carbon dioxide heat exchangers. However, for these initial simulations we used the small scale geometry described in the previous section. Fig. 13 shows the temperature variations at both the sink and source sides for Lead-Bismuth alloy layer heated from below. A sinusoidal temperature variation, given by equation (6), was applied to source side of the convective system. It is easy to observe (Fig. 13) that Lead-bismuth also follows the characteristics of convective system discussed previously.

$$T_{source} = 673 + 10\sin(0.1t) \quad (6)$$

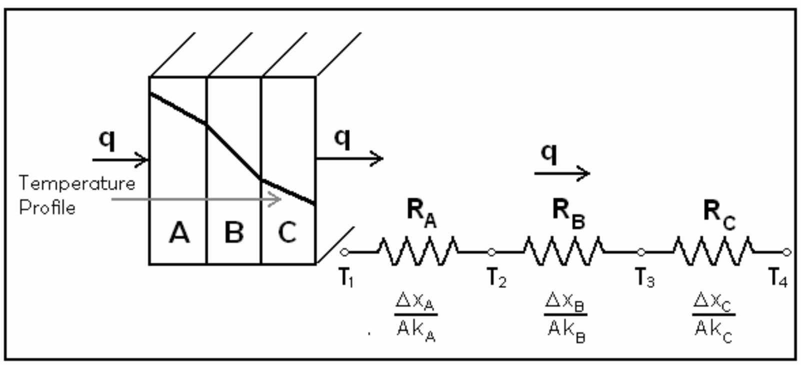

Fig. 1. Equivalent circuit for conduction.

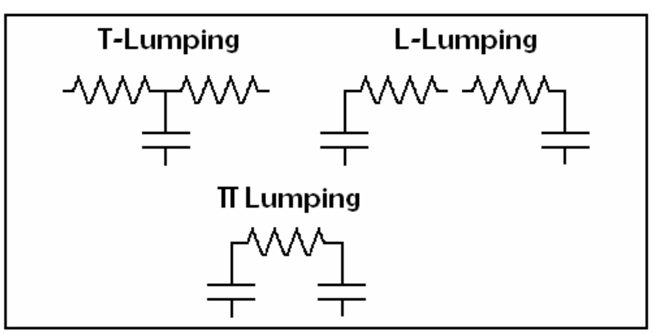

Fig. 2. Lump systems configurations.

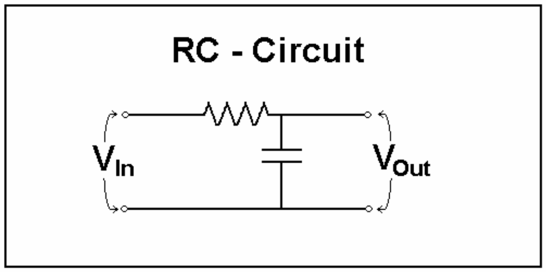

Fig. 3. Integrator circuit.

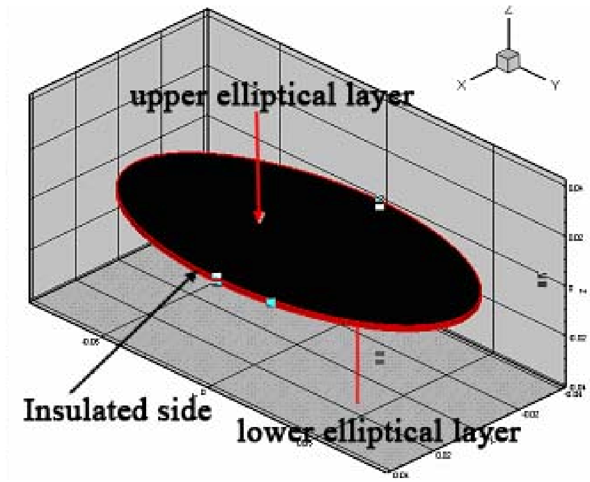

Fig. 4. Plot of the elliptical fluid volume investigated.

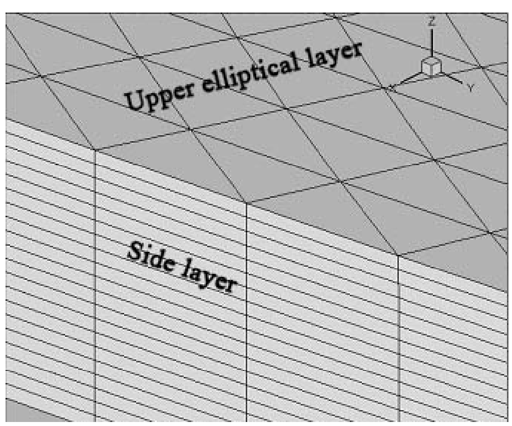

Fig. 5. The computational grid for the elliptical fluid volume.

V. CONCLUSIONS

The proposed analogy[12] between RC electrical circuit and conduction and convection heat transfer systems is tested using FLUENT numerical simulations. Various characteristics of an RC electrical circuit are compared to the characteristics of conductive and convective heat transfer systems.

Conduction system (heated from top) was proved to exhibit the filtering characteristics very much like an RC integrator circuit. High frequency fluctuations at the source side are completely filtered and do not appear at the sink side, where as low frequency fluctuations are permitted through the conductive system. Our new simulation demonstrated that a convective system (heated from bottom) also exhibit similar filtering characteristics.

Transmission of the amplitude of fluctuations at the source side was also investigated. It was observed that analogous to RC circuits both conductive and convective systems cause amplitude attenuation. Transmission coefficient at the sink side for convective system was greater than that for conductive system based on the fact that convection enhances heat transfer. From the RC electrical circuit point of view this is because of the different material properties and flow characteristics of the two systems which results in different characteristic time constants for both systems.

During this study we also demonstrated that the characteristic time constant of a heat transfer system does not depend on the temperature difference between the two plates. This result is also consistent with the analogy between an integrator circuits where the circuit characteristic time constant is independent of the step voltage applied.

Finally our initial simulations with Lead-bismuth eutectic alloy show that convective system with metallic coolant also behaves very similar to RC electric circuit.

VI. ACKNOWLEDGMENTS

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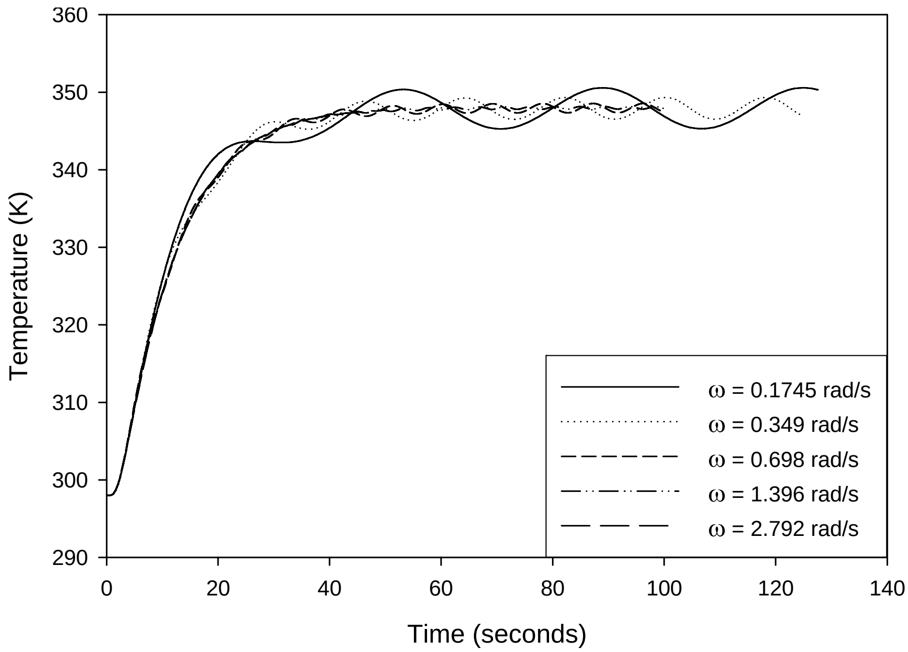

Fig. 6. Sink side temperature response for a pure conductive system when a sinusoidal variation in temperature is applied at the source side. The effect of source side frequency is obvious from these results.

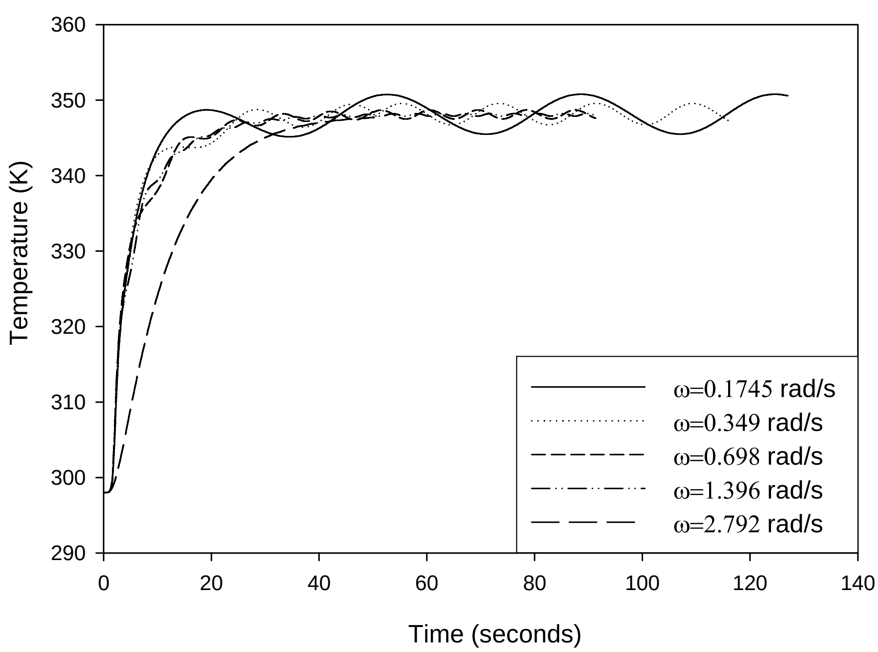

Fig. 7. Sink side temperature response for convective system when a sinusoidal variation in temperature is applied at the source side. The effect of source side frequency is obvious from these results.

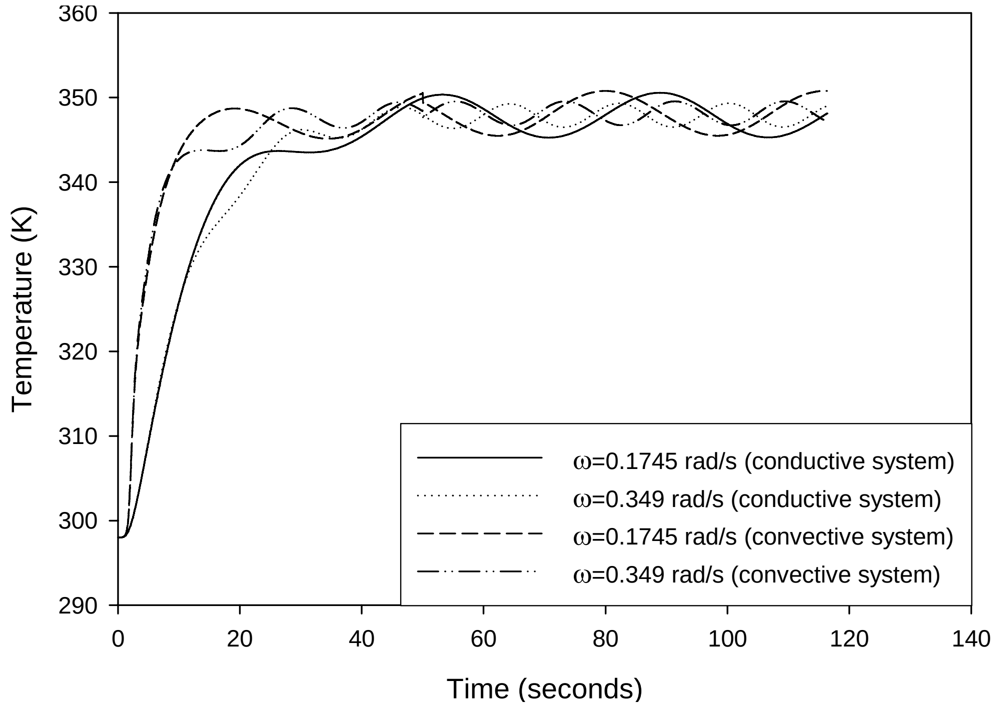

Fig. 8. Comparison between conductive and convective system response to a sinusoidal temperature variation at the source side and non convective boundary conditions at the sink side.

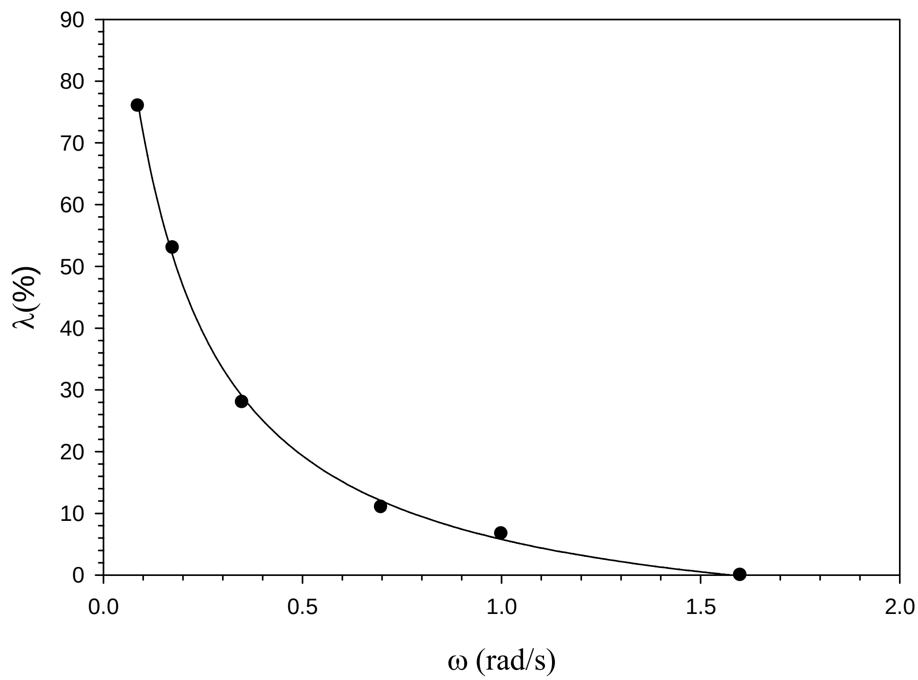

Fig. 9. Transmission factor vs. frequency (circular) of temperature oscillation for conductive system with non-convective boundary conditions. Convective system with non-convective boundary conditions produced identical results.

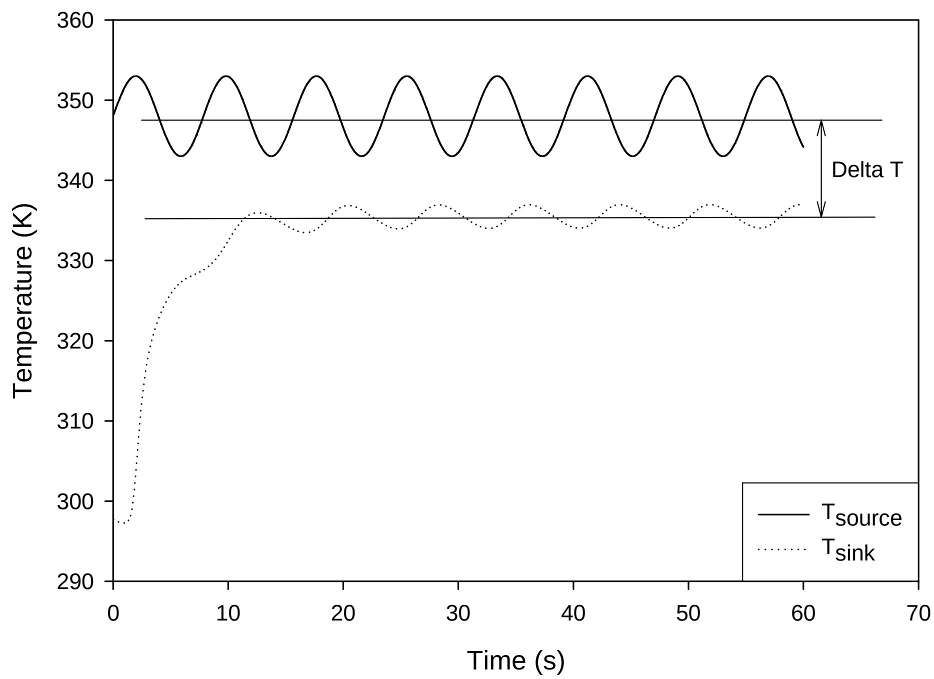

Fig. 10. Temperature variations at sink and source sides for the case of bottom plate heating. Frequency of oscillations is 0.8 rad/s.

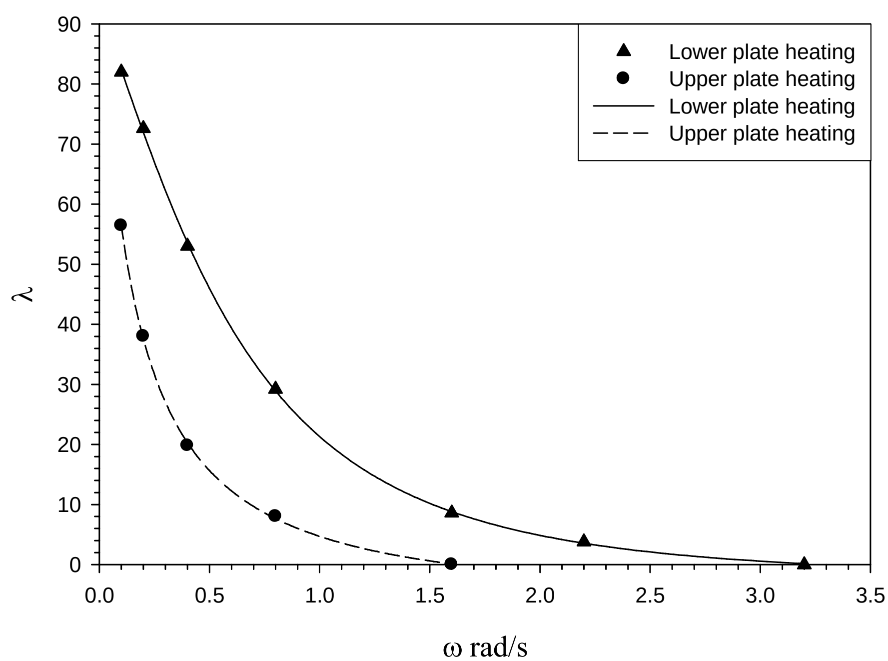

Fig. 11. Transmission factor vs. frequency (circular) of temperature oscillation for both conductive and convective systems with convective boundary conditions.

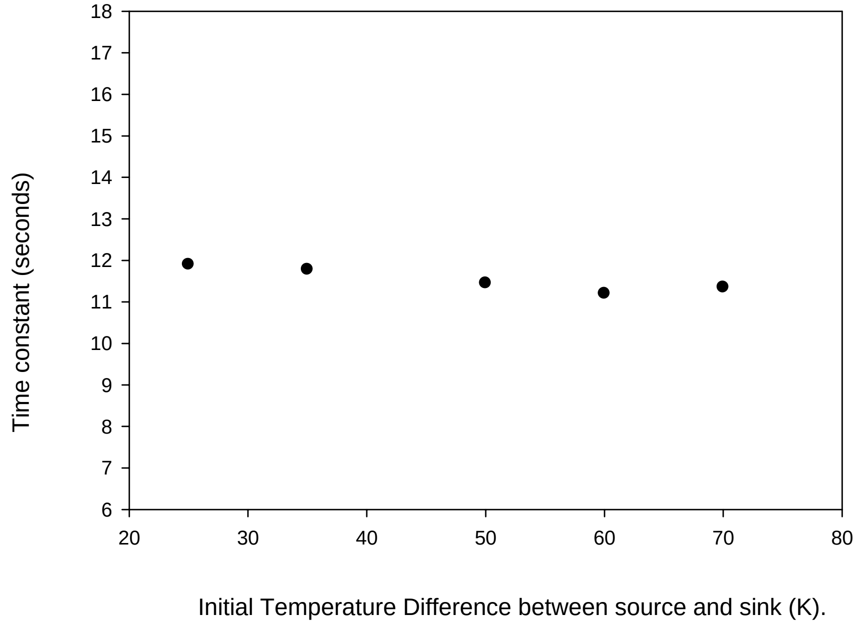

Fig. 12. Applied temperature gradient vs. observed characteristic time constant.

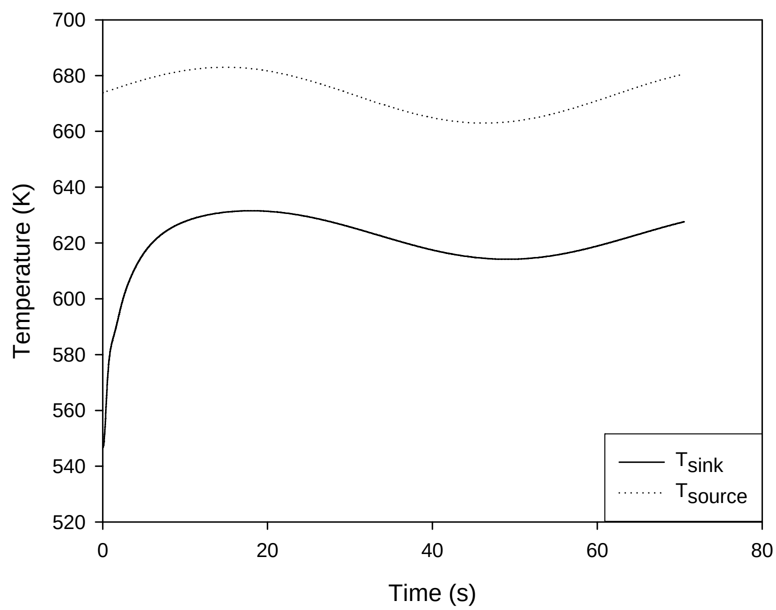

Fig. 13. Source and sink temperature variations for a convective system with Lead-bismuth as working fluid. Sinusoidal temperature variations are applied to the source side.

NOMENCLATURE

Symbol	**Description**	**Units**
A	Layer cross sectional area	m^2
A_o	Amplitude of oscillations	°K
A_{sink}	Amplitude of oscillations at sink	°K
A_{source}	Amplitude of oscillations at source	°K
C	Electrical Capacitance	F
k	Thermal Conductivity	$W.m^{-1}.^oK^{-1}$
q	Heat Flux	$W\ m^2$
R	Electrical Resistance	Ω
t	Time	S
T_{source}	Temperature at source side	^oK
T_{sink}	Temperature at sink side	^oK
V_{In}	Input Voltage	V
V_{Out}	Output Voltage	V
Δx	Material Thickness	M
GREEK		
α	Thermal Diffusivity	$M^2.s^{-1}$
θ	Difference between sink temperature at any time t and initial sink	^oK
θ_0	Difference between source temperature and initial sink temperature	^oK
ν	Kinematic viscosity	$M^2.s^{-1}$
ρ	Density	$Kg.m^{-3}$
τ	Characteristic Time Constant	S
ω	Circular frequency	rad/s
λ	Transmission factor	

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PAPER II

Explicit Inversion Formulas for Tridiagonal Matrices

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Abstract

Explicit formulas for the elements of the inverse of tridiagonal matrices are developed. The formulas are recursive and applicable to symmetric, non-symmetric, equal and non-equal coefficient matrices. For the case of a general tridiagonal matrix four recursive formulas are developed to obtain the elements of the matrix inverse. All formulas are deduced based on the fact that a matrix multiplied by its inverse result in a unit matrix. Also three special cases are discussed. First, for a symmetric tridiagonal matrix the elements of the matrix inverse are obtained from two recurrence formulas instead of four. Second, for a tridiagonal Toeplitz symmetric matrix two simple recurrence formulas are deduced to obtain the elements of the matrix inverse. Third, for a general constant diagonal matrix (Toeplitz) four recurrence formulas are derived to obtain the elements of the matrix inverse. Finally, the results are applied to matrices arising from discretization of two-point boundary value problems. These results are very helpful in economical simulation of reactor physics, heat and mass transport, electrical circuit solution, fluid flow and many other engineering problems.

Keywords: *Tridiagonal* matrix inverse; Symmetric tridiagonal matrix inverse; Tridiagonal Toepletz matrix inverse

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1. Introduction

Tridiagonal matrices and block tridiagonal (scalar Penta-diagonal) matrices arise in many applied sciences and engineering applications 1-3. Tridiagonal linear system appears in the solution of parabolic partial differential equations, finite difference solution of boundary value problems and construction of cubic splines. Different methods are available to obtain the inversion of tridiagonal matrices. Direct expressions for the inversion of tridiagonal matrices arising from boundary value problems are well established 3-5. Also, Indirect formulas for the inversion of tridiagonal matrices were developed 1, 6-8. Explicit formula for the inverse of tridiagonal Toeplitz matrix was developed 9. Recently an explicit method was developed for the elements of the inverse of a general tridiagonal matrix [10].

In the present study we derive explicit recursive formulas for inversion of nonsingular tridiagonal matrices. The recursive formulas were derived based on the rule that a matrix multiplied by its inverse will yield an identity matrix. To obtain simple recurrence formulas for the inverse of general tridiagonal matrices, the tridiagonal matrix is first decomposed to two matrices whose multiplication product is the tridiagonal matrix. The first matrix is a diagonal matrix and the second matrix is a tridiagonal matrix with the subdiagonal elements equal to one. For the tridiagonal matrix with subdiagonal elements equal to one, a recursive formula was deduced to obtain the elements of the first column of the matrix inverse. Consequently, a second recursive formula was deduced to obtain the elements of the lower part of the matrix inverse ($i \geq j$). A third recursive formula was also deduced to obtain the elements of the last column of the matrix inverse. Finally, a recursive formula is deduced to obtain the rest of the elements of the upper part of the inverse matrix ($i < j$). In the case of a symmetric tridiagonal matrix the recursive formulas becomes simpler and only two formulas are deduced to obtain the elements of the matrix inverse. One is used for the elements of the first column and another formula is used for the rest of the elements of the lower part of the matrix. The upper part of the inverse matrix is then obtained simply based on the fact that a symmetric tridiagonal matrix inverse is also symmetric. In the case of symmetric tridiagonal Toeplitz matrix the recursive formulas become simpler and two formulas are deduced to obtain the elements of the matrix inverse.

2. Analysis

Let A be an nxn real non-singular tridiagonal matrix

$$A = \begin{bmatrix} \alpha_1^* & \beta_1^* & 0 & \cdots & 0 & 0 & 0 \\ \gamma_2^* & \alpha_2^* & \beta_2^* & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \gamma_{n-1}^* & \alpha_{n-1}^* & \beta_{n-1}^* \\ 0 & 0 & 0 & \cdots & 0 & \gamma_n^* & \alpha_n^* \end{bmatrix} \qquad (1)$$

We define:

$$\alpha_1 = \alpha_1^* \qquad (2a)$$

$$\beta_1 = \beta_1^* \qquad (2b)$$

$$\alpha_i = \alpha_i^* / \gamma_i^* \text{ , } i\text{=2, 3,….., } n \qquad (2c)$$

$$\beta_i = \beta_i^* / \gamma_i^* \text{ , } i\text{=2, 3,….., } n \qquad (2d)$$

Where:

$$\gamma_i^* \neq 0 \qquad (2e)$$

To obtain a simple recurrence formula for the inversion of matrix A, we rewrite the matrix A as multiplication of a diagonal matrix D and a matrix C with coefficients of the lower diagonal equal to unity as follows:

$$A = \begin{bmatrix} \alpha_1^* & \beta_1^* & 0 & \cdots & 0 \\ \gamma_2^* & \alpha_2^* & \beta_2^* & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \gamma_n^* & \alpha_n^* \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \gamma_2^* & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \gamma_n^* \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ 1 & \alpha_2 & \beta_2 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ 0 & \cdots & 1 & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & \cdots & 1 & \alpha_n \end{bmatrix} = DC$$

The inverse of matrix A is

$$A^{-1} = C^{-1} D^{-1} \tag{3a}$$

or

$$A^{-1} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ 1 & \alpha_2 & \beta_2 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ 0 & \cdots & 1 & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & \cdots & 1 & \alpha_n \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1/\gamma_2^* & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1/\gamma_n^* \end{bmatrix} \tag{4}$$

Denote

$$C^{-1} = \frac{1}{d} C_{i,j} \tag{5}$$

Where $C_{i,j}$ are the elements of the matrix inverse (i and j represent the row and column respectively) multiplied by a constant d (we will drive a formula for d later).

Assuming that C is a 4x4 tridiagonal matrix and C^{-1} is its inverse, we write equation (6).

$$C \ x \ C^{-1} = I$$

$$\begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ 1 & \alpha_2 & \beta_2 & 0 \\ 0 & 1 & \alpha_3 & \beta_3 \\ 0 & 0 & 1 & \alpha_4 \end{bmatrix} \left[\frac{1}{d} \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} \\ C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} \\ C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} \\ C_{4,1} & C_{4,2} & C_{4,3} & C_{4,4} \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

We determine the elements of the first column in C^{-1} by setting the last element of the column to be one

$$C_{n,1} = C_{4,1} = 1 \ , \tag{7}$$

then by multiplying the n^{th} row (4^{th} row in this case) in matrix C by the first column of the inverse matrix, one determines the n-1 element (3^{rd} element) as follows:

$$\begin{aligned} &C_{3,1} + \alpha_4 C_{4,1} = 0 \\ &C_{3,1} = -\alpha_4 C_{4,1} = -\alpha_4 \end{aligned} \tag{8}$$

then we multiply the 3^{rd} row in matrix C by the first column of the inverse matrix to determine the 2^{nd} element of the inverse matrix as follows:

$$\begin{aligned} &C_{2,1} + \alpha_3 C_{3,1} + \beta_3 C_{4,1} = 0 \\ &C_{2,1} = -\alpha_3 C_{3,1} - \beta_3 C_{4,1} \end{aligned} \tag{9}$$

The process is repeated by multiplying the n-1 row (3^{rd} row) in matrix C by the first column of the inverse matrix to determine the n-2 element (2^{nd} element). This process has the following recurrence formula:

$$C_{i,1} = -\beta_{i+1} C_{i+2,1} - \alpha_{i+1} C_{i+1,1} \tag{10}$$

For i=n-1,…….., 1

Where:

$$\beta_n = 0$$

After generating the elements of the first column, we determine the rest of the inverse matrix elements, $C_{i,j}$, from the first column elements $C_{i,1}$ as follows:

For $i \geq j$ or the lower part of the inverse matrix:

To get the element $C_{4,2}$ we multiply the last row of the inverse matrix by the first column of the matrix C as follows:

$$\begin{aligned} &\alpha_1 C_{4,1} + C_{4,2} = 0 \\ &C_{4,2} = -\alpha_1 C_{4,1} \end{aligned} \tag{11}$$

To get the element $C_{4,3}$ we multiply the last row of the inverse matrix by the second column of the matrix C as follows:

$$\begin{aligned} &\beta_1 C_{4,1} + \alpha_2 C_{4,2} + C_{4,3} = 0 \\ &C_{4,3} = -\alpha_2 C_{4,2} - \beta_1 C_{4,1} \end{aligned} \tag{12}$$

To obtain the rest of the elements of the lower part of the inverse matrix we continue this process which has the following recurrence relation:

$$C_{i,j} = -\alpha_{j-1} C_{i,j-1} - \beta_{j-2} C_{i,j-2} \tag{13}$$

For: $j = m$, $i = n, n\text{-}1, n\text{-}2, \ldots.., m$

$m = 2, 3, 4, \ldots., n$

Where:

$\beta_0 = 0$

The lower part of the matrix becomes:

$$Lp\, C^{-1} = \frac{1}{d}\begin{bmatrix} C_{1,1} & \bullet & \bullet & \bullet \\ (\alpha_3\alpha_4 - \beta_3) & (\alpha_3\alpha_4 - \beta_3)(-\alpha_1) & \bullet & \bullet \\ -\alpha_4 & (\alpha_4)(\alpha_1) & (-\alpha_1\alpha_2 + \beta_1)\alpha_4 & \bullet \\ 1 & (-\alpha_1) & (\alpha_1\alpha_2 - \beta_1) & C_{4,4} \end{bmatrix} \tag{14}$$

Where:

$C_{1,1} = \alpha_2(-\alpha_3\alpha_4 + \beta_3) + \beta_2\alpha_4$ (From equation 10)

$C_{4,4} = -\alpha_3(\alpha_1\alpha_2 - \beta_1) - \beta_2(-\alpha_1)$ (From equation 13)

Now we complete the elements of the upper part of the inverse matrix. First, we try to get a formula for the first element in the last column. This will make the analysis simpler as we will see.

Multiplying the 3rd row in the matrix C^{-1} by the 4th column of the matrix C we get equation 15. Multiplying the 3rd row in matrix C by the 4th column in C^{-1} we get equation 16. Multiplying the 2nd row in C by the 4th column in C^{-1} we get equation 17.

$$C_{3,4} = -\frac{\beta_3}{\alpha_4} C_{3,3} \tag{15}$$

$$C_{2,4} = -\beta_3 C_{4,4} - \alpha_3 C_{3,4} \tag{16}$$

$$C_{1,4} = -\beta_2 C_{3,4} - \alpha_2 C_{2,4} \tag{17}$$

Solving equation 15, 16 and 17 together we get equation 18 which is a formula for the first element in the last column of the inverse matrix. In an *n*x*n* matrix this formula will take the form shown in equation 19.

$$C_{1,4} = \beta_1\beta_2\beta_3 \tag{18}$$

$$C_{1,n} = \beta_1\beta_2 ....\beta_{n-1} = \prod_{i=1}^{n-1} \beta_i \qquad (19)$$

and the rest of the elements of the last column are generated the same way we generated the elements of the first column and we write the following recurrence formula:

$$C_{i,n} = \frac{-C_{i-2,n} - \alpha_{i-1} C_{i-1,n}}{\beta_{i-1}} \qquad (20)$$

For: i=2, 3, …., n-1

Where: $C_{0,n} = 0$

Following exactly the previous procedures we get the recurrence formula for the upper half of the inverse matrix C^{-1} ($i < j$):

$$C_{i,j} = -\left(\frac{\alpha_{j+1}}{\beta_j}\right) C_{i,j+1} - \left(\frac{1}{\beta_j}\right) C_{i,j+2} \qquad (21)$$

For: j=m and i=m-1, m-2,…., 1

m=n-1, n-2, ….., 2

Where: $C_{i,n+1} = 0$

The upper half of the inverse matrix Up C^{-1} is

$$Up\ C^{-1} = \frac{1}{d}\begin{bmatrix} \bullet & (-\beta_3 + \alpha_3\alpha_4)\beta_1 & (-\alpha_4\beta_1\beta_2) & (\beta_1\beta_2\beta_3) \\ \bullet & \bullet & (\alpha_1\alpha_4\beta_2) & (-\alpha_1\beta_2\beta_3) \\ \bullet & \bullet & \bullet & (-\beta_1 + \alpha_1\alpha_2)\beta_3 \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \qquad (22)$$

To get the constant, d, we multiply the first row in the matrix C by the first column of the inverse matrix C^{-1} (which is equals to 1) and we obtain the following formula for d:

$$d = \alpha_1 C_{1,1} + \beta_1 C_{2,1} \qquad (23)$$

that is $d = -\alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_4\beta_2 + \alpha_1\alpha_2\beta_3 + \alpha_3\alpha_4\beta_1 - \beta_1\beta_3$

Finally, the inverse of matrix A is obtained from equation (4).

Special Case (1): Inverse for symmetric tridiagonal matrix

The recursive formulas to obtain the elements of the inverse of the tridiagonal matrix becomes very simple and reduces only to two simple recursive formulas (one for the first column elements and the other for the rest of the elements of the inverse matrix) in the case of a symmetric tridiagonal matrix (as the one shown in equation (16)). We do not need to decompose the matrix to two matrices as we did previously.

$$C = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & 0 & \cdots & 0 \\ 0 & \beta_2 & \alpha_3 & \beta_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & 0 & \cdots & \beta_{n-1} & \alpha_n \end{bmatrix} \tag{24}$$

Following the same steps discussed previously and based on the fact that a matrix multiplied by its inverse is equal to a unit matrix the elements of the inverse matrix could be found as follows:

Elements of the first column are obtained as follows:

$$C_{n,1} = 1 \tag{25}$$

$$C_{i,1} = \frac{-\alpha_{i+1} C_{i+1,i} - \beta_{i+1} C_{i+2,1}}{\beta_i} \tag{26}$$

For: $i=n-1, n-2, \ldots\ldots\ldots, 1$

Where: $\beta_n = 0$

Comparing equation (26) with equation (10) we found that both are the same but in case of equation (10) the matrix had $\beta_i = 1$.

The rest of the elements of the inverse matrix are obtained using the following recurrence formula:

For $i \geq j$:

$$C_{i,j} = \frac{-\beta_{j-2} C_{i,j-2} - \alpha_{j-1} C_{i,j-1}}{\beta_{j-1}} \tag{27}$$

For: $j=m$ and $i=n, n-1, n-2, \ldots, m$

m=2, 3, ……., n

Where: $C_{i,0} = 0$ and $\beta_0 = 0$

Comparing equation (13) and equation (27) we find out that both are the same, but in equation (13) $\beta_{j-1} = 1$.

Based on the fact that the tridiagonal matrix was symmetric this will cause its inverse to be also symmetric and the rest of the elements are obtained from:

For $i < j$:

$$C_{i,j} = C_{j,i} \tag{28}$$

Finally, all the elements of the inverse matrix ($C_{i,j}$) should be multiplied by the constant $1/d$ which is obtained from equation (23).

Special Case (2): Inverse for tridiagonal symmetric constant diagonal (Toeplitz) matrix

If the tridiagonal matrix takes the form of a symmetric Toeplitz matrix (equation (29)) the recursive formula to obtain the elements of the inverse matrix becomes even simpler.

$$C = \begin{bmatrix} \alpha & \beta & 0 & \cdots & 0 \\ \beta & \alpha & \beta & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \beta & \alpha & \beta \\ 0 & 0 & \cdots & \beta & \alpha \end{bmatrix} \tag{29}$$

Assume that the elements of the inverses matrix are $C_{i,j}$ where i and j are the row and column of the inverse matrix respectively.

Using the same technique discussed previously the elements of the first column of the inverse matrix are obtained as follows:

$$C_{n,1} = 1 \tag{30}$$

$$C_{i,1} = \frac{-\alpha C_{i+1,1} - \beta C_{i+2,1}}{\beta} \tag{31}$$

For: $i=n-1, n-2, \ldots\ldots\ldots, 1$

Where:

$$C_{n+1,1} = 0$$

The rest of the elements of the lower half of the inverse matrix are obtained using the following recurrence formula:

For $i \geq j$:

$$C_{i,j} = C_{i+1,j-1} + C_{i-1,j-1} - C_{i,j-2} \qquad (32)$$

For: $j=m$, $i=n, n-1, n-2, \ldots, m$

$m=2, 3, \ldots.., n$

Where:

$$\left(C_{i,0}\right), \left(C_{0,j}\right), \left(C_{n+1,j}\right) = 0,$$

For $i < j$:

$$C_{i,j} = C_{j,i} \qquad (33)$$

Finally, all the elements of the inverse matrix $(C_{i,j})$ should be multiplied by a constant $1/d$ which is obtained from equation (23).

Special Case (3): Inverse for tridiagonal matrix with constant diagonals (Toeplitz)

Consider a case where the tridiagonal matrix has constant diagonals (Toeplitz). The matrix takes the form shown in equation (34).

$$C = \begin{bmatrix} \alpha & \beta & 0 & \cdots & 0 \\ \gamma & \alpha & \beta & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \gamma & \alpha & \beta \\ 0 & 0 & \cdots & \gamma & \alpha \end{bmatrix} \qquad (34)$$

In this matrix the super diagonal, diagonal and sub diagonal are constant and equal to β, α and γ respectively. It is well known that a Toeplitz matrix is persymmetric and so, if it exists, is its inverse. Again we will start the analysis without decomposing the matrix into two.

The elements of the first column are obtained exactly as we mentioned previously.

First column elements:

$$C_{n,1} = 1 \tag{35}$$

$$C_{i,1} = -\frac{\alpha}{\gamma}C_{i+1,1} - \frac{\beta}{\gamma}C_{i+2,1} \tag{36}$$

For: i=n-1, n-2, ….., 1

Where $C_{n+1,1} = 0$

Elements of lower part of the inverse matrix ($i \geq j$):

We only need to determine the elements on and below (or above) the antidiagonal (Inverse matrix is persymmetric). They are obtained as follows:

$$C_{i,j} = -\frac{\alpha}{\gamma}C_{i,j-1} - \frac{\beta}{\gamma}C_{i,j-2} \tag{37}$$

For: j=m and i=n-m+1, n-m, n-m-1,….., m

Where: m=2, 3, 4, …., K

$$K = \frac{n+1}{2} \text{ if } n \text{ is odd or } K = \frac{n}{2} \text{ if } n \text{ is even}$$

$$C_{i,0} = 0$$

The rest of elements of the lower matrix are obtained according to:

$$C_{i,j} = C_{n-j+1,n-i+1} \tag{38}$$

For $i \geq j$

Elements of the Last Column:

In this matrix the sub diagonal elements are not equal to one. Trying to deduce a formula for the first element in the last column will be complex. Any element obtained in the last column will be sufficient to develop the recurrence formula and substitute in it. We will find the element n-1 in the last column as follows:

Multiplying the n-1 row in the C matrix by the n column of the inverse matrix we get:

$$C_{n-1,n} = -\frac{\beta}{\gamma}C_{n-1,n-1} \tag{39}$$

The following recurrence formula is developed to obtain the rest of the elements of the last column of the inverse matrix:

$$C_{i,n} = \frac{-\beta C_{i+2,n} - \alpha C_{i+1,n}}{\gamma} \tag{40}$$

For: $i=n-2, n-3, \ldots\ldots, 1$

For the Upper part of the inverse matrix ($i < j$):

$$C_{i,j} = -\left(\frac{\alpha}{\beta}\right) C_{i,j+1} - \left(\frac{\gamma}{\beta}\right) C_{i,j+2} \tag{41}$$

For: $j=m$ and $i=n-m+1, n-m+2, \ldots.., m-1$

Where: $m= n-1, n-2, \ldots., K+1$

$$C_{i,n+1} = 0$$

The rest of the elements of the upper part of the inverse matrix are obtained from:

$$C_{i,j} = C_{n-j+1,n-i+1} \tag{42}$$

For: $i < j$

Example:

The 3x3 matrix defined in equation (43) was used by Mallik [10] as an example.

$$I = \begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 \\ 0 & \gamma_3 & \alpha_3 \end{bmatrix} \tag{43}$$

Decomposing into two matrices:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ 1 & \alpha_2/\gamma_2 & \beta_2/\gamma_2 \\ 0 & 1 & \alpha_3/\gamma_3 \end{bmatrix} \tag{44}$$

$$I^{-1} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ 1 & \alpha_2/\gamma_2 & \beta_2/\gamma_2 \\ 0 & 1 & \alpha_3/\gamma_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\gamma_2 & 0 \\ 0 & 0 & 1/\gamma_3 \end{bmatrix} \tag{45}$$

The inverse matrix is then obtained by direct substitution in equations 7, 10, 13, 19, 20, 21, 23 and 45 as shown in equation (46). After a number of steps Mallik [10] arrived at the same inverse of the matrix. The proposed method can be used to considerably reduce the effort to arrive at the inverse.

$$I^{-1}=\frac{1}{(\alpha_1\alpha_2\alpha_3-\alpha_3\beta_1\gamma_2-\alpha_1\beta_2\gamma_3)}\begin{bmatrix}\alpha_2\alpha_3-\beta_2\gamma_3 & -\alpha_3\beta_1 & \beta_1\beta_2\\ -\alpha_3\gamma_2 & \alpha_1\alpha_3 & -\alpha_1\beta_2\\ \gamma_2\gamma_3 & -\alpha_1\gamma_3 & \alpha_1\alpha_2-\beta_1\gamma_2\end{bmatrix}\quad(46)$$

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